Static critical behavior in the inactive phase of the pair contact process

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Steady-state properties in the absorbing phase of the $1d$ pair contact process model are investigated. It is shown that, in typical absorbing states (reached by the system's dynamic rules), the density of isolated particles ρ_1, approaches a stationary value that depends on the annihilation probability (p); the deviation from its "natural" value at criticality ρ_1^{nat} follows a power law: $\rho_1^{nat}-\rho_1\sim(p-p_c)^{\beta_1}$ for $p>p_c$. Monte Carlo simulations yield $\beta_1=0.81$. A cluster approximation is developed for this model, qualitatively confirming the numerical results and predicting $\beta_1=1$. The singular behavior of the isolated particles density in the inactive phase is explained using a phenomenological approach.

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I. INTRODUCTION

In the simplest models undergoing absorbing state phase transitions in the directed percolation (DP) universality class, such as the contact process (CP), the stationary state of the system in the inactive phase is the state devoid of particles. Other models in the same class, similar to the pair contact process (PCP), have however a richer structure, associated with the existence of an infinite number of absorbing states [1,2]. In the inactive phase, the absorbing state the system evolves to depends on the initial conditions and the distance from the critical point, and so does the average density of isolated particles in the stationary state. In the case of PCP, the field responsible for the dynamics (the density of pairs of particles ρ_2) is coupled to another field (the density of isolated particles ρ_1). This background of isolated particles is responsible for the nonuniversality of some dynamic properties of the system at criticality [3,4]. Recently, the one-dimensional PCP with particle diffusion (known as PCPD or annihilation/fission model) has received a lot of attention and was at the center of some controversy [5,6,8,7,9]; nevertheless, its critical behavior is not yet fully clarified. Carlon *et al.* [6] presented a pair approximation study of this model that however is not appropriate for the nondiffusive case. When diffusion is absent, all activity stops as soon as the number of pairs vanishes, because isolated particles are immobile. In contrast, initially isolated particles may give rise to a pair if diffusion is allowed. This explains why the expected behavior of PCP is qualitatively different from the $D=0$ limit of PCPD. In a proper treatment of PCP, *isolated particles* have to be counted separately from the total number of particles.

In PCP, as the system approaches the critical point from the active phase, the nonordering field approaches a "natural" value (ρ_1^{nat}), and its behavior is described by the same power laws as those of the order parameter [10].

In the present paper, we investigate the behavior of this model in the inactive phase by looking at properties of its *natural absorbing states*, the absorbing states selected by the system's dynamics. As shown by both mean-fieldlike approximations and Monte Carlo (MC) simulations, the stationary density of isolated particles develops a power-law singularity as it approaches the critical point.

II. PHENOMENOLOGICAL APPROACH

In the PCP, nearest-neighbor pairs of particles annihilate each other with probability p or create, with probability $1-p$, a particle at one of the adjacent (vacant) sites to the pair. In the inactive phase, the concentration of pairs is known to decay exponentially: $\rho_2(t)\sim e^{-\alpha t}$, with $\alpha\sim(p-p_c)^{\nu_{||}}$. The exponent $\nu_{||}$ is associated with the temporal correlation length and mean-field theory predicts $\nu_{||}=1$.

Now, if one is interested in studying the stationary single-particle concentration as a function of $p(>p_c)$, one may refer to the coupled Langevin equations describing this dynamical process. These equations are constructed to reflect the symmetry of the problem and envolve some phenomenological parameters that are related in an indirect way to the microscopic dynamic process under study (see [11] for details). At the mean-field level, the time evolution of ρ_1 is given by

$$\frac{d\rho_1}{dt}=r_1\rho_2-w_1\rho_1\rho_2-u_1\rho_2^2, \tag{1}$$

obviously coupled to the evolution of ρ_2. With the change of variables $\overline{\rho_1}=\rho_1-r_1/w_1$, Eq. (1) reads

$$\frac{d\overline{\rho_1}}{dt}=-w_1\rho_2\overline{\rho_1}-u_1\rho_2^2. \tag{2}$$

In the following, we will assume $\rho_2(t)=\rho_2(0)e^{-\alpha t}$. Then, with the change of variables $t_1=e^{-\alpha t}$, Eq. (2) may be solved exactly

$$\overline{\rho_1}(t)=-\frac{u_1\alpha}{w_1^2}\left(\frac{w_1\rho_2(0)}{\alpha}e^{-\alpha t}+1\right)+Ce^{w_1e^{-\alpha t}\rho_2(0)/\alpha}, \tag{3}$$

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with

$$C=\left(\overline{\rho_1}(0)+\frac{u_1}{w_1}\rho_2(0)+\frac{u_1\alpha}{w_1^2}\right)e^{-w_1\rho_2(0)/\alpha}. \tag{4}$$

In the limit $t\to\infty$, one has

$$\overline{\rho_1}(t\to\infty)=-\frac{u_1\alpha}{w_1^2}+C. \tag{5}$$

In the vicinity of the critical point, when $\alpha\to 0$, C decreases more rapidly than α, and one may therefore conclude that

$$\overline{\rho_1}(t\to\infty)\simeq-\frac{u_1\alpha}{w_1^2}. \tag{6}$$

At the critical point, $\overline{\rho_1}(t\to\infty)=0$, i.e.,

$$\rho_1(t\to\infty)=\frac{r_1}{w_1}. \tag{7}$$

ρ_1^{nat} is then given by r_1/w_1, as already found in [10]. But now we can see that $\rho_1(t\to\infty)$ varies with p, in the inactive phase

$$\rho_1(t\to\infty)\simeq\rho_1^{nat}-\frac{u_1}{w_1^2}\alpha. \tag{8}$$

We must point out that the present argument relies on a mean-field approach and on the assumption that $\rho_2(t)=\rho_2(0)e^{-\alpha t}$ (which may not be true at early times). A variation of $\rho_1(t\to\infty)$ with p is then plausible. However, the prediction that the critical exponent of the quantity $\rho_1^{nat}-\rho_1$ should be equal to $\nu_{||}$ is only valid within a mean-field approximation and is not expected to apply to the one-dimensional (1D) PCP model.

III. CLUSTER APPROXIMATION

Mean-fieldlike kinetic equations for the PCPD were obtained in [6]. Whereas the singlesite approximation is only appropriate to the high-diffusion limit ($D\to 1$), the pair approximation gives a good qualitative picture of the model for $D>0$, but does not show some important characteristics of the PCP without diffusion. Indeed, according to the pair approximation, the steady-state single-particle density vanishes at the critical point ($p_c=0.2$ for the pair approximation with $D=0$); however, it is well established that, at the critical point, the single-particle density approaches a nonzero value, ρ_1^{nat} ($\rho_1^{nat}\simeq 0.242$, in a sequential dynamical process). Also, the pair approximation predicts a power-law temporal decay for the pair concentration in the inactive phase, contrary to the exponential decay found in the simulations.

In the coarse-grained Langevin description, the fields that are used to characterize the system configurations are the local pair density and the local density of isolated particles. If one uses two-site clusters, as in the pair approximation, configurations with isolated particles are not treated appropriately. Indeed, if one aims at building a better approximation, one has to go to three-site clusters and consider P_{010}, the probability of having an isolated particle at the center of the cluster. The price to pay is that of increasing complexity: the number of variables and equations increases and these have to be solved numerically; Carlon *et al.* dealt with two variables and two equations, so they were able to obtain some results analytically. We have used a (3,2)-cluster approximation [12] whose technical details are given in the Appendix.

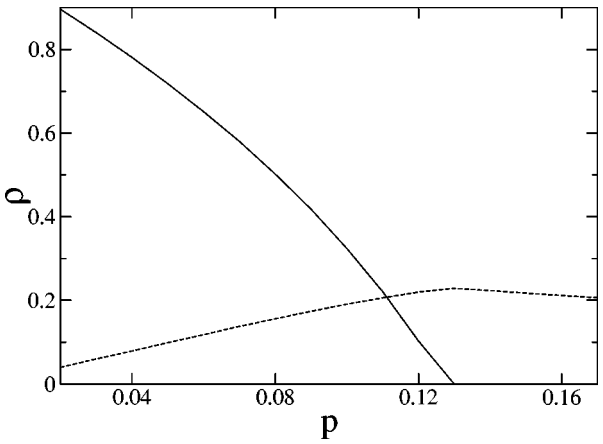

FIG. 1. Pair density ρ_2 (–) and density of isolated particles ρ_1 (– – –) as a function of p, within the (3,2)-cluster approximation.

In Fig. 1, we have plotted the stationary values of P_{11} and P_{10} as functions of p. As shown, the concentration of pairs vanishes for $p\geqslant 0.128$, i.e., $p_c=0.128$ within the present approximation. The estimate obtained by MC simulations is $p_c=0.0771$; so, as expected, the present result is an improvement when compared to what was obtained by the pair approximation. More interestingly, the concentration of particles at the critical point ($\rho_1=P_{10}$, since $P_{11}=0$) is nonzero; indeed, $\rho_1=0.229$, a value not far from $\rho_1^{nat}=0.2418(2)$, as obtained by the simulations [13]. On the other hand, a linear decrease of ρ_1 for $p>p_c$ may also be noticed.

IV. NUMERICAL SIMULATIONS

In this paper, we concentrate on the critical behavior of ρ_1 in the inactive phase. The simulations that we present were done on systems of size $L=5000$; to ensure a sufficient number of pairs, we chose an initial particle concentration of 0.5. The numbers of particles and of pairs were recorded versus time up to a number of Monte Carlo steps, t_{max}, which ranged from $t_{max}=10^6$ for p very close to the transition ($p=0.0775$) down to $t_{max}=10^4$ for $p>0.085$. These times were chosen such that most of the samples (typically around 20 000) had already entered the absorbing state. For each p, we evaluated the average activity time τ_{av}. To obtain the stationary value of ρ_1, we have averaged the final number of particles in those samples that entered the absorbing state at a time around τ_{av}. The short-lived samples were excluded in order to eliminate finite-size effects. On the other hand, in

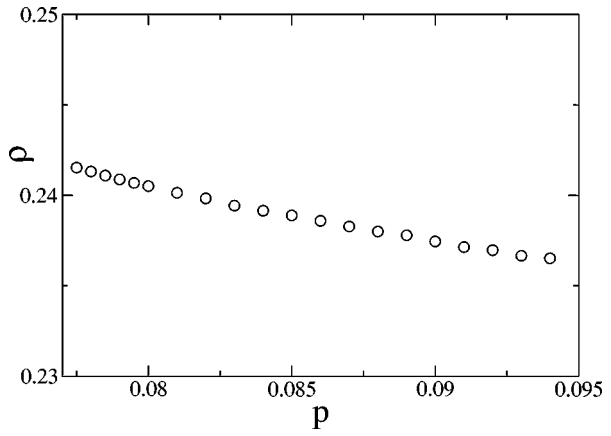

FIG. 2. Stationary particle density (in the inactive phase) as a function of p.

systems of this size, and for p close to p_c, the concentration of pairs is still considerably high (typically around 0.1) at very long times—thus, the concentration of isolated particles is lower than in samples where the number of pairs is vanishingly small. For this reason, we have also ignored the contribution from these long-lived samples. We have chosen to consider samples with $0.75\tau_{av}<\tau<1.25\tau_{av}$, but have checked that the results are stable with respect to other choices in a reasonable range.

In Fig. 2, we show the stationary single-particle concentration as a function of p. Figure 3 shows a log-log plot of $(\rho_1^{nat}-\rho_1)$ vs $(p-p_c)$, with $\rho_1^{nat}=0.2418$ and $p_c=0.0771$. A linear fit of the data is clearly appropriate and leads to an exponent $\beta_1=0.81(3)$. This seems to be a different exponent, not simply related to the DP exponents that describe the critical behavior of other quantities, in the active phase of the PCP model.

V. CONCLUSION

In this paper, we show that the approach to the critical point in the in inactive phase of the 1D PCP model is signaled by a power-law singularity of a static quantity—the deviation of the particle concentration from its critical-point value ρ_1^{nat}. A cluster-mean-field approximation was developed that predicts a finite value of ρ_1^{nat}, in reasonable agreement with MC simulations and a linear decrease of ρ_1 with $(p-p_c)$; this is confirmed by a phenomenological approach. Such a decrease in ρ_1 is also shown in our MC simulations; the best fit to the data is consistent with an exponent $\beta_1=0.81(3)$, different from the mean-field prediction.

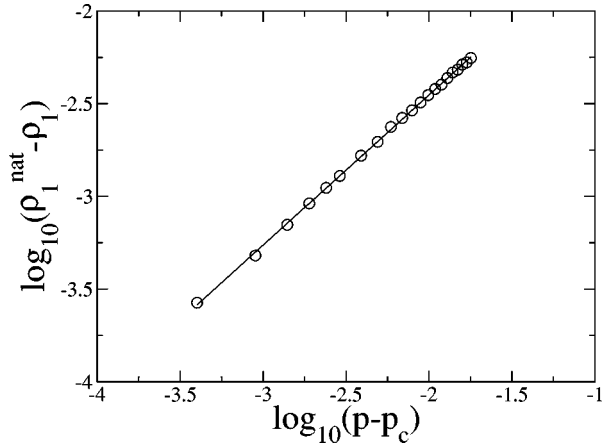

FIG. 3. Log-log plot of $(\rho_1^{nat}-\rho_1)$ vs $(p-p_c)$, with $\rho_1^{nat}=0.2418$ and $p_c=0.0771$. The straight line is a least-squares linear fit, with slope=0.81.

Critical behavior of a static quantity in the inactive phase has been observed by Lipowski and Droz [14] in a rather different model with an infinite number of absorbing states. In that case, a random walk argument leads to a relationship between the corresponding exponent and the order-parameter exponent β. In the case of PCP, such an argument cannot be applied and whether β_1 is related to the (DP) critical exponents that characterize the active phase of PCP requires further investigation. The same applies to other systems with infinitely many absorbing states.

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APPENDIX

In the (3,2)-cluster approximation [12] one uses three-site clusters and allows for an overlap of two sites between adjacent clusters. Thus, the probability of a six-site cluster in the state $ABCDEF$ is written as

$$P_{ABCDEF}=P_{ABC}\frac{P_{BCD}}{P_{BC}}\frac{P_{CDE}}{P_{CD}}\frac{P_{DEF}}{P_{DE}}. \tag{A1}$$

We use $x\equiv P_{111}$, $y\equiv P_{110}(=P_{011})$, $z\equiv P_{100}(=P_{001})$, $v\equiv P_{101}$ as independent variables. It can be easily shown that $P_{010}=v+z-y$, and $P_{000}=1-x-y-2v-3z$. We then study all the processes that may occur when a pair belonging to a six-site cluster is selected.

Take, for example, the configuration 1 1 1 1 1 1. According to the above approximation, the probability of this configuration is $A_1=x^4/c^3$, where $c=P_{11}=x+y$. When the central pair is selected, then, with rate p, the configuration 1 1 0 0 1 1 is generated. The variation in the number of three-site clusters in the state 1 1 1 is then $\Delta x=-4$; analogously, the changes in the number of three-site clusters in configurations 1 1 0, 1 0 0, and 1 0 1 are, respectively, $\Delta y=+1, \Delta z=+1, \Delta v=0$.

On the other hand, according to the PCP dynamic rules, a pair may also create a particle at a randomly chosen nearest neighbor provided this is vacant. If one considers the configuration 1 $\underline{1\ 1}$ 0 1 1 (whose probability is given, within the present approximation, by $A_{11}=xy^2v/cd^2$, with $d\equiv P_{10}=v+z$), then, with rate $1-p$, the configuration

1 1 1 1 1 1 is generated. This change corresponds to $\triangle x=+3, \triangle y=-1, \triangle z=0, \triangle v=-1$.

The kinetic equations for x,y,z,v are obtained by considering all the contributions from all the possible six-site configurations

$$\frac{dx}{dt}=p[-4A_1-6A_2-4A_3-2A_4-4A_6-2A_7-2A_9]+(1-p)[3A_{11}+A_{12}+3A_{13}+2A_{14}+A_{15}+A_{16}+2A_{17}+A_{18}],$$

$$\frac{dy}{dt}=p[A_1-A_4-A_5-2A_7-2A_8-2A_9-A_{10}]+(1-p)[-A_{11}-A_{13}],$$

$$\frac{dz}{dt}=p[A_1+2A_2+2A_3+A_4+A_5+2A_7-A_{10}]+(1-p)[-A_{12}-A_{15}],$$

$$\frac{dv}{dt}=p[-2A_3-2A_5-2A_7-2A_8]+(1-p)[-A_{11}+A_{12}-A_{13}-A_{14}+A_{15}-A_{17}],$$

where

$$A_1=\frac{x^4}{c^3},\quad A_2=\frac{x^3y}{c^3},\quad A_3=\frac{x^2vy}{c^2d},\quad A_4=\frac{x^2y^2}{c^3},$$

$$A_5=\frac{y^2v^2}{cd^2},\quad A_6=\frac{x^2yz}{c^2d},\quad A_7=\frac{xy^2v}{c^2d},\quad A_8=\frac{y^2zv}{cd^2},$$

$$A_9=\frac{xy^2z}{c^2d},\quad A_{10}=\frac{y^2z^2}{cd^2},\quad A_{11}=\frac{xy^2v}{cd^2},$$

$$A_{12}=\frac{xyz^2}{cd(1-c-2d)},\quad A_{13}=\frac{y^3v}{cd^2},$$

$$A_{14}=\frac{xyv(v+z-y)}{cd^2},\quad A_{15}=\frac{y^2z^2}{cd(1-c-2d)},$$

$$A_{16}=\frac{xyz(1-c-2d-z)}{cd(1-c-2d)},\quad A_{17}=\frac{y^2v(v+z-y)}{cd^2},$$

$$A_{18}=\frac{y^2z(1-c-2d-z)}{cd(1-c-2d)}.$$

These equations were solved numerically.

[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1996).

[2] For a recent review, see H. Hinrichsen, Adv. Phys. **49**, 1 (2000).

[3] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993); I. Jensen and R. Dickman, Phys. Rev. E **48**, 1710 (1993).

[4] J.F.F. Mendes, R. Dickman, M. Henkel, and M.C. Marques, J. Phys. A **27**, 3019 (1994).

[5] M.J. Howard and U.C. Taüber, J. Phys. A **30**, 7721 (1997).

[6] E. Carlon, M. Henkel, and U. Schollwöck, Phys. Rev. E **63**, 036101 (2001).

[7] G. Ódor, Phys. Rev. E **62**, R3027 (2000).

[8] H. Hinrichsen, Phys. Rev. E **63**, 036102 (2001).

[9] K. Park, H. Hinrichsen, and I. Kim, Phys. Rev. E **63**, 065103 (2001).

[10] G. Ódor, J.F. Mendes, M.A. Santos, and M.C. Marques, Phys. Rev. E **58**, 7020 (1998).

[11] M.A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, Phys. Rev. Lett. **76**, 451 (1996); Physica D **103**, 485 (1997).

[12] D. ben-Avraham and J. Kohler, Phys. Rev. A **45**, 8358 (1992).

[13] R. Dickman, W.R.M. Rabelo, and G. Ódor, e-print cond-mat/0108438.

[14] A. Lipowski and M. Droz, Phys. Rev. E **64**, 031107 (2001).